OPTIMAL DESIGN OF HYBRID COMPOSITE LAMINATED PLATES UNDER INPLANE LOADING

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CERTIFICATE

This is to certify that the work entitled [illegible] of SWEPT LAMINATED PLATES UNDER IMPLANE LOADING, has been carried out under my supervision and has not been submitted elsewhere [illegible].

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SYMBOLS AND NOTATIONS

a	:	Length of the plate in x-direction
b	:	Length of the plate in y-direction
N	:	Number of plies in a laminate
$E_1, E_2, G_{12}, \nu_{12}$	:	Elastic constants of an orthotropic lamina
E_f, G_f, ν_f	:	Elastic constants of fibre material
E_m, G_m, ν_m	:	Elastic constants of matrix material
Q_{ij}	:	Reduced Stiffnesses
$\overline{\sigma_1}, \overline{\sigma_2}, \overline{\sigma_{12}}$	:	Inplane Stresses
$\epsilon_1, \epsilon_2, \epsilon_{12}$	:	In-plane strain components
$\overline{Q}_{ij}(\theta)$	:	Transformed reduced stiffnesses
θ_i	:	Fibre orientation in i-th lamina
t_i	:	Thickness of i-th lamina
ρ_i	:	Density of i-th lamina
A_{ij}, B_{ij}, D_{ij}	:	Elastic constants of a laminate plate
z_i	:	Distance of i-th lamina from the geometric mid plane of the laminate
R_1	:	t_1/t_3 is the thickness ratio of a six layered laminate
R_2	:	t_2/t_3 is the thickness ratio of a six layered laminate
$2t$	:	Total thickness of a laminate
λ_{cr}	:	Critical buckling load factor

u_o, v_o	:	Mid-plane displacements in x and y directions respectively
w	:	Lateral displacement
$(\epsilon_x^o, \epsilon_y^o, \epsilon_{xy}^o)$	:	Mid plane strains in the xy plane
(k_x, k_y, k_{xy})	:	Curvatures in the xy plane
N_x, N_y, N_{xy}	:	Resultant forces acting on a laminate
M_x, M_y, M_{xy}	:	Resultant moments acting on a laminate
$\bar{N}_x$, $\bar{N}_y$, $\bar{N}_{xy}$	:	Applied in plane forces per unit width
ϕ	:	Buckling load $= \frac{a^2 \lambda_{cr}}{t^3}$

ABSTRACT

In the present work analytical expressions are used, which calculate thickness ratios of the layers of a given six layered antisymmetric angle ply laminate, to make it decoupled. Ensuring decoupling, increases the stiffness and strength of the laminate, simplifies the analysis of thin laminated plates and saves computation efforts considerably.

One of the objective, of this thesis is to maximize critical buckling load of the hybrid laminated plates under inplane loading. Hybrid laminate considered for this study is the hybrid laminate made of one type of fibres only but with different fibre packings in each layer. The design variables are fibre orientations and volume fractions in each ply. Total thickness of the laminated plate is assumed to be specified. Computations were carried out on Boron-epoxy hybrid system for various aspect ratios, loading combinations and boundary conditions.

Optimization study on hybrid laminates made of two types of fibres is also made. Computations are carried out on carbon-glass,- epoxy composite system for, carbon/carbon/glass anti-symmetric, six-layered laminate. The objective function is to minimize cost for a specified load. Total thickness of the plate is assumed specified simply supported and clamped boundary conditions have been considered.

The buckling analysis is carried out using Galerkin's method. Exterior penalty function method along with Powell's method has been used for constrained maximization.

The results are presented in tabular and graphical forms. It is observed that hybrid laminates made of one type of fibres only, have as high as 25 percent extra buckling strength than the homogeneous laminates, for the same given cost and weight. Optimum volume fractions and percentage thickness of laminae have been found to be insensitive to aspect ratios, loading combinations and boundary conditions. In case of carbon-glass hybrid system, carbon/carbon/glass combination has wide range of loads, above the load possible by glass hybrid and at lower range of loads at least 20 - 25 per cent cost can be saved by using such hybrid laminates as compared to carbon hybrid laminates.

CHAPTER I

INTRODUCTION AND LITERATURE SURVEY

1.1 INTRODUCTION:

For the past few years, fibre reinforced composite laminates have found wide applications in primary structures of aircrafts and spacecrafts. Due to their high strength to weight ratio and stiffness to weight ratio, they have attracted much attention and are being used to replace the conventional materials. Anisotropic properties of composite laminates can be suitably exploited. By imparting directional properties to the laminates, specific requirements can be met, henceforth overall weight of the structure can be reduced considerably. Their efficient utilization needs judicious selection of various design variables such as fibre orientations, thicknesses of laminae and stacking sequence etc. Hence a purpose oriented study and experimentation on composite laminates has created an area of scientific research which has continuously updated itself with time.

Most of the reported research work considers homogeneous laminates. Laminates made of two or more types of fibres, called hybrid laminates are very popular because of their certain improved properties over the laminates made of identical fibres. Hybrid laminates are prepared, by stacking the laminae of different fibres and of different properties. They offer many advantages, such as excellent specific stiffness and strength, toughness, fatigue life and wear resistance. Another distinctive

advantage of hybrid systems is that they provide a practical means to achieve intermediate properties/cost levels. Hybrids often have the properties, which neither of their constituents possess. Hybrids need not necessarily have more than one type of fibres. Composites having same fibre but varying fibre packing in each ply, behave as if made of different materials. All these advantages associated with hybrid laminates make them very suitable for advanced structural applications.

Unlike isotropic materials, composites exhibit a wide variety of elastic couplings like shear extension coupling or bending-stretching coupling. Such couplings cause unwanted warping of laminates and make their analysis difficult. Their presence reduces strength and stiffness of laminates considerably. It is not possible to eliminate coupling completely for any arbitrary stacking sequence of the laminae. However, manipulating thicknesses, fibre orientation and stacking sequence can reduce or even completely eliminate the detrimental effects of couplings. Some studies have been made to achieve laminates free of couplings.

The advantages of high strength to weight ratio, durability, ease of repair, design flexibility, high energy absorptivity and seamless construction associated with composite laminated plates, offer an attractive alternative to metallic aircraft structural components.

The common practical situation encountered is thin plates subjected to inplane forces. Judicious selection of various

design parameters is required to achieve improved designs to carry maximum possible buckling load by thin laminated plates. Hence suitable optimization techniques can be applied to achieve superior design together with overcoming the difficulty of selection of design variables available. Generally, the design variables are function of geometry and material properties. Therefore, in a model optimization situation the design variables should be selected carefully to utilize full potential of composites.

1.2 LITERATURE SURVEY:

Amhartsumyan [1] recognized the effect of coupling between inplane stretching and plate bending in 1953. This coupling effect is the major difference between the elastic response of an arbitrarily laminated plates and homogeneous plates.

Werren and Norris [2] advocated that for the laminates containing laminae of identical orthotropic properties and thicknesses, it is possible to orient the layers in such a way that the resultant elastic stretching behaviour is isotropic. Accofding to them to achieve this result the total number of layers must be three or more and a typical K-th layer of a n - layered laminate must be oriented at an angle $\theta_K = \pi(K-1)/n$. However, this laminate, in general, is not isotropic with regard to bending and stretching-bending coupling.

Bartholomew [3] suggested that for uncoupled, orthotropic laminates, made of laminae of identical thicknesses and material properties, it is possible to achieve orthotropy in bending as well as in-plane by selection of a suitable ply stacking order. The fibres lie symmetrically about the orthotropic axes of the laminate.

Sharma, Iyengar and Murty [4] have shown that it is possible to eliminate the coupling effects in four and six layered antisymmetric crossply laminates by a suitable choice of thickness ratios of the laminae. Coupling effects are completely eleminated for few six-layered angle-ply laminates by a suitable combination of stacking sequence and thickness ratios of laminae. Thickness ratios are shown to be independent of material properties.

Jones [5] used classical laminated plate theory including the Kirchoff's hypothesis to analyze buckling and dynamic response of unsymmetrically laminated cross-ply rectangular plates simply supported on all sides.

Holston [6] analyzed the buckling response of orthotropic plates with three edges simply supported and the remaining edge free by developing an approximate equation for the buckling load.

Paul, Jenson and Finch [7] examined the effects of inherent mechanical couplings exhibited in fully anisotropic (i.e. unsymmetric) graphite/epoxy laminates on the buckling

loads and mode shapes. The results indicate that the couplings, especially those which relate stretching and bending behaviour cause out of plane deflections prior to buckling and reduce the buckling load significantly. The analytical results also show that the mode-shapes exhibit twisting due to mechanical couplings.

Klaus Rohwer [8] has analyzed plate with different boundary conditions, edge ratios and fibre angles under uniaxial as well as biaxial normal stresses to see the bending-twisting coupling effects on the buckling load of symmetrically stacked plates. An indicator is established which allows to approximately determine the buckling load reduction with the aid of the stiffness coefficients alone. For plate under compression it allows to decide prior to the buckling analysis whether or not it is necessary to include coupling.

Soni [9] obtained a minimum weight design of laminated plate under transverse and dynamic loading, subject to multiple constraints. Fibre orientation and thickness of each ply were treated as design variables. Optimization was carried out by employing penalty function method.

Hirano [10] maximized buckling load treating fibre angles as design variables for orthotropic angle ply laminated plates simply supported on all sides. He, however, considered thicknesses of all the laminae identical.

Joshi [11] carried out the optimization studies on simply supported antisymmetric angle ply and orthotropic angle ply laminated plates, subjected to biaxial compression and transverse loading. The behaviour constraints are on stability strength and the deflection of the laminate.

Umaretiya [12] considered hybrid rectangular and skewed antisymmetric angle ply laminated plates, simply supported on two opposite sides and clamped on the remaining. He obtained the optimization results for minimum weight, treating individual ply thickness and fibre orientations as design variables.

Barwey [13] has made an attempt to achieve specially orthotropic behaviour of antisymmetric cross ply and angle ply hybrid composite laminates taking fibre orientations and thicknesses as design variables. He has also maximized critical buckling load of laminated plates. He has shown that at optimum point laminate tends to decouple.

Study of the available literature reveals that in optimization studies of laminates under buckling, fibre volume fractions of plies are kept constant and have not been considered as design variables.

The advantage of simplified analysis of laminated plates by ensuring decoupling where possible, has not been incorporated in the optimization study. Hybrid laminates made of one type of fibres with different fibre packing in each ply have not been studied.

The purpose of the present study is to fill the existing gaps as far as possible. The following section gives a brief account of the work undertaken and an overall view of the present thesis.

1.5 SCOPE OF PRESENT WORK:

Coupling behaviour present in composite laminates have degrading effects on their strength and stiffness. Hence for the buckling strength of a laminate to be maximum, it must be completely free from all couplings. It has been reported by Barwey [13] that at the optimum point, laminates tend to decouple. For the coupled laminates, it is required to solve three coupled equations in u, v, and w, the displacements along the three axis, to compute buckling load for a given laminate. This, not only needs complicated analysis but also involves considerable efforts in computation. Since in a optimization process these computations have to be repeated several times, it becomes very time consuming and costly process. If it is possible to make the laminate decoupled, one needs to consider only one governing differential equation instead of three. This makes analysis very simple. In the present work, this fact is taken care of, by ensuring decoupling at each step of optimization process. In the optimization process design variables are constrained to assume only these value which make laminate decoupled. Iyenger [4] has obtained completely decoupled laminate for few six layered antisymmetric angle stacking sequences. In the present work analytical expressions

are used which ensure complete decoupling in six-layered antisymmetric laminates. Ensuring decoupling at each step of optimization makes analysis simpler, shows faster convergence and hence computation time is saved considerably.

In the Chapter II of the present work, expressions are given to generate laminae properties from fibre and matrix properties. The governing differential equation for buckling of rectangular laminated plates are presented. The boundary conditions considered are:

1. All sides simply supported
2. All sides clamped
3. Two opposite sides simply supported and other two clamped

Analytical expressions are given which ensure decoupling in six layered antisymmetric angle ply laminate. Brief theory of optimization method used is also given.

In Chapter III, optimization study is made on special class of hybrid laminates, made of one type of fibres. Buckling load is maximized for specified total fibre content. Design variables are fibre volume fractions and fibre orientations. All the three boundary conditions given above are considered.

In Chapter IV, hybrid laminates made of two types of fibres are dealt. Optimization study is made on carbon/carbon/glass antisymmetric six layered laminate. Overall cost is

minimized for specified intermediate loads, and compared with that of pure carbon hybrid laminate. Design variables are fibre volume fraction and fibre orientations of each ply. Simply supported and clamped boundary conditions have been considered.

CHAPTER II

ANALYSIS AND PROBLEM FORMULATION

A composite laminate is constructed by stacking several thin laminae over each other. The properties of the laminate, depend upon the properties of each layer, stacking angle sequence and thicknesses. A composite laminate exhibits a variety of couplings. Such couplings in additions to complicating the analysis of laminated plates result in an overall reduction in the stiffness, of the laminate. Many attempts have been made in this direction. As reviewed in the first chapter, investigators have obtained decoupling for cross ply laminates and also for few other angle stacking sequences, by selecting suitable thicknesses of the laminae. In this chapter, analytical expressions are given to eliminate couplings completely for the case of six layered antisymmetric angle ply laminate. These expressions calculate thicknesses of the laminae of a laminate of given material properties and stacking sequence to eradicate couplings. Both hybrid and homogeneous laminates have been considered.

Empirical relations to generate composite properties from fibre and matrix properties, are given. Governing equations for thin laminated plates and also their simplification for the case of decoupled laminate are given in this chapter. Brief discription of the optimization method used is given at the end of the chapter.

2.1 MATERIAL PROPERTIES:

The properties of a composite laminate depend on the properties of its constituents i.e. fibres and matrix materials, their distribution and some other factors. Theoretical and semiempirical methods of determining composite properties are used to predict the effects of a large number of variables like fibre and matrix properties and their volume fractions etc. In the present work composite properties have been generated using the formulae given in [14]

$$E_1 = E_f V_f + E_m (1 - V_f) \tag{2.1a}$$

$$E_2 = \frac{1 + \Psi V_f \left[\frac{E_f/E_m - 1}{E_f/E_m + \Psi}\right]}{1 - V_f \left[\frac{E_f/E_m - 1}{E_f/E_m + \Psi}\right]} \tag{2.1b}$$

where Ψ is a constant. Here, $\Psi = 2$.

$$G_f = \frac{E_f}{2(1 + \nu_f)}$$

$$G_m = \frac{E_m}{2(1 + \nu_m)} \tag{2.2}$$

$$G_{12} = \frac{1 + \Psi V_f \left[\frac{G_f/G_m - 1}{G_f/G_m + \Psi} \right]}{1 - V_f \left[\frac{G_f/G_m - 1}{G_f/G_m + \Psi} \right]} \qquad (2.3)$$

Here, constant $\Psi = 1$,

$$\nu_{12} = \nu_f V_f + \nu_m (1 - V_f) \qquad (2.4)$$

These composite properties, now can be used to generate laminate properties.

2.2 ANALYSIS:

Consider a N-layered laminate as shown in Fig. 2.1. Each layer made of continuous fibre reinforced material, is considered to be orthotropic with uniform material properties E_1, E_2, G_{12} and ν_{12}, where subscripts 1 and 2 refer to the axis along and perpendicular to the fibre directions respectively as shown in Fig. 2.2. Hence for such laminate, stress strain relationship of a typical K-th layer is expressed as [15]:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_{12} \end{Bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}_k \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} \qquad (2.5)$$

where Q_{ij}s, called reduced stiffnesses, are

$$
\begin{aligned}
Q_{11} &= E_1/(1 - \nu_{12}\nu_{21}) \\
Q_{12} &= \nu_{21} E_1/(1 - \nu_{12}\nu_{21}) \\
&= \nu_{12} E_2/(1 - \nu_{12}\nu_{21}) \\
Q_{66} &= G_{12} \\
Q_{22} &= E_2/(1 - \nu_{12}\nu_{21})
\end{aligned}
\tag{2.6}
$$

The poisson's ratio ν_{21} can be obtained from the reciprocal relation

$$\nu_{12}/E_1 = \nu_{21}/E_2 \tag{2.7}$$

If the fibres in a lamina lie at an angle θ to the X-axis of reference plane, Fig. 2.3, then the transformed stiffness matrix for the K-th lamina is given by,

$$\bar{Q}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \tag{2.8}$$

where,

$$
\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta \\
\bar{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta) \\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta \\
\bar{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta + (Q_{12}-Q_{22}+2Q_{66})\sin^3\theta\cos\theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12}-Q_{22}+2Q_{66})\sin\theta\cos^3\theta \\
\bar{Q}_{66} &= (Q_{11}+Q_{22}-Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta)
\end{aligned}
\tag{2.9}
$$

From the above relations, it is clear that if fibres are oriented at $-\theta$ to X-axis and having identical material properties, all the elements $\bar{Q}_{ij}$ remain same except $\bar{Q}_{16}$ and $\bar{Q}_{26}$, which differ only in sign.

Mathematically,

$$(\bar{Q}_{ij})_{-\theta} = -(\bar{Q}_{ij})_{+\theta} \quad , \qquad i = 1,2,\ j = 6$$

and $$(\bar{Q}_{ij})_{-\theta} = (Q_{ij})_{+\theta} \quad , \text{ for all other values of } i \text{ and } j \tag{2.10}$$

Using Kirchoff Love's thin plate hypothesis [16], with usual notations, the following constitutive relations are obtained:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \left[\begin{array}{ccc|ccc} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ \hline B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{array}\right] \begin{Bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \epsilon_{xy}^o \\ K_x \\ K_y \\ K_{xy} \end{Bmatrix} \tag{2.11}$$

Here the extensional coupling and bending stiffnesses which are expressed as A_{ij}, B_{ij} and D_{ij} respectively, are

$$A_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k \, (Z_k - Z_{k-1}) \tag{2.12a}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} (\bar{Q}_{ij})_k \, (Z_k^2 - Z_{k-1}^2) \tag{2.12b}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (\bar{Q}_{ij})_k \, (Z_k^3 - Z_{k-1}^3) \tag{2.12c}$$

2.3 GOVERNING EQUATIONS FOR RECTANGULAR LAMINATED PLATES:

The laminated plate analysis is based on the following assumptions [15]:

(a) Each layer is orthotropic, linear elastic and of constant thickness.

(b) Thickness of the plate is very small compared to its length and width.

(c) Body forces are absent.

(d) The displacements u, v and w in x,y and z directions respectively are small compared to the plate thicknesses.

(e) Strains ϵ_x, ϵ_y and ϵ_{xy} are small compared to unity.

(f) The plate is in a state of plane stress.

(g) Transverse shear deformations are neglected.

With these assumptions the governing differential equation for the plate buckling are obtained as [15]:

$$A_{11} u_{o,xx} + 2A_{16} u_{o,xy} + A_{66} u_{o,yy} + A_{16} v_{o,xx} + (A_{12}+A_{66}) v_{o,xy} + A_{26} v_{o,yy} - B_{11} w_{,xxx} - 3B_{16} w_{,xxy} - (B_{12}+2B_{66}) w_{,xxx} - B_{26} w_{,yyy} = 0 \quad (2.13)$$

$$A_{16} u_{o,xx} + (A_{12}+A_{66}) u_{o,xy} + A_{26} u_{o,yy} + A_{66} v_{o,xx} + 2A_{26} v_{o,xy} + A_{22} v_{o,yy} - B_{16} w_{,xxx} - (B_{12}+2B_{66}) w_{,xxy} - 3B_{26} w_{,xyy} - B_{22} w_{,yyy} = 0 \quad (2.14)$$

$$D_{11} w_{,xxxx} + 4D_{16} w_{,xxxy} + 2(D_{12}+2D_{66}) w_{,xxyy}$$
$$+ 4D_{26} w_{,xyyy} + D_{22} w_{,yyyy} - B_{11} u_{o,xxx} - 3B_{16} u_{o,xxy}$$
$$- (B_{12} + 2B_{66}) u_{o,xyy} - B_{26} u_{o,yyy} - B_{16} v_{o,xxx}$$
$$- (B_{12} + 2B_{66}) v_{o,xxy} - 3B_{26} v_{o,xyy} - B_{22} v_{o,yyy}$$
$$= - \bar{N}_x w_{,xx} - 2\bar{N}_{xy} w_{,xy} - \bar{N}_y w_{,yy} \qquad (2.15)$$

In above eqns. A_{ij}s, B_{ij}s and D_{ij}s are defined as in eqns. (2.12). The quantities $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ are the applied inplane loads per unit width as shown in Fig. 2.4. The inplane loads are expressed in terms of common load factor λ as,

$$\bar{N}_x = \lambda K_1$$
$$\bar{N}_y = \lambda K_2 \qquad (2.16)$$
$$\bar{N}_{xy} = \lambda K_3$$

Using these relations the eqn. (2.15) reduces to,

$$-B_{11} u_{o,xxx} - 3B_{16} u_{o,xxy} - (B_{12}+2B_{66}) u_{o,xyy}$$
$$-B_{26} u_{o,yyy} - B_{16} v_{o,xxx} - (B_{12}+2B_{66}) u_{o,xxy}$$
$$-3B_{26} v_{o,xyy} - B_{22} v_{o,yyy} + D_{11} w_{,yyyy}$$
$$+ 4D_{16} w_{,xxxy} + 2(D_{12}+2D_{66}) w_{,xxyy}$$
$$+ 4D_{26} w_{,xyyy} + D_{22} w_{,yyyy}$$
$$= - \lambda (K_1 w_{,xx} + K_2 w_{,yy} + 2K_3 w_{,xy}) \qquad (2.17)$$

These equations are coupled in u, v and w. It is observed that since load terms appear in eqn. (2.17) only, it can be

solved independently for calculating λ if B_{ij}s are made zero. Hence, for a decoupled laminate for which coupling matrix is zero, one needs to solve only single governing equation and thus analysis in considerably simplified.

2.4 DECOUPLING ANALYSIS:

Consider a six layered antisymmetric angle ply laminate as shown in Fig. 2.5. In such a laminate for every lamina oriented atan angle θ above the midplane there exists an identical lamina placed at an equal distance below the midplane but oriented at a negative angle θ. From the definitions of B_{ij} and D_{ij} as given in eqn. (2.12), it is clearly seen that for antisymmetric laminate,

$$B_{11} = B_{12} = B_{22} = B_{66} = D_{16} = D_{26} = 0 \quad (2.18)$$

One therefore needs to make B_{16} and B_{26} zero for decoupling between inplane displacements u, v and transversed displacement w.

For given material properties and fibre orientations of the laminae, it may be possible to make B_{16} and B_{26} zero by suitably selecting the thickness ratios of the layers of a laminate.

Thickness ratios, R_1 and R_2 are defined as

$$R_1 = t_1/t_3$$
$$R_2 = t_2/t_3 \quad (2.19)$$

In case of hybrid laminates these thickness ratios depend on material properties and fibre orientations, to make E_{16} and E_{26} zero. Defining

$$\alpha_{ij} = \frac{(\bar{Q}_{ij})_2}{(\bar{Q}_{ij})_1} \quad , \qquad \beta_{ij} = \frac{(\bar{Q}_{ij})_3}{(\bar{Q}_{ij})_1} \tag{2.20}$$

where, i = 1,2 and j = .6 and subscripts 1,2 and 3 represent respective laminae. Defining

$$\begin{aligned} A_k &= (Q_{11} - Q_{12} - 2Q_{66})_k \\ B_k &= (Q_{12} - Q_{22} + 2Q_{66})_k \\ F_k &= \sin\theta_k \cos^3\theta_k, \\ G_k &= \sin^3\theta_k \cos\theta_k \end{aligned} \tag{2.21}$$

where subscript k represents respective laminae. Using these quantities thickness ratios R_2 and R_1 to eliminate coupling are determined from the relations,

$$\begin{aligned} R_2 &= \sqrt{1+\gamma} - 1 \\ R_1 &= -\sqrt{1+\gamma} + \sqrt{(1+\gamma(1-\alpha_{16}) - \beta_{16})} \end{aligned} \tag{2.22a}$$

where,

$$\gamma = \frac{\beta_{26} - \beta_{16}}{\alpha_{16} - \alpha_{26}} \tag{2.22b}$$

$$\alpha_{16} = \frac{A_2 F_2 + B_2 G_2}{A_1 F_1 + B_1 G_1} \; ; \; \beta_{16} = \frac{A_3 F_3 + B_3 G_3}{A_1 F_1 + B_1 G_1}$$

$$\alpha_{26} = \frac{A_2 G_2 + B_2 F_2}{A_1 G_1 + B_1 F_1} \; ; \; \beta_{26} = \frac{A_3 G_3 + B_3 F_3}{A_1 G_1 + B_1 F_1} \tag{2.23}$$

Now γ can be rewritten as,

$$\gamma = \frac{(A_1A_3 - B_3B_1)(F_1G_3 - G_1F_3) + (A_1B_3 - A_3B_1)(F_1F_3 - G_1G_3)}{(A_2A_1 - B_2B_1)(F_2G_1 - G_2F_1) + (A_2B_1 - B_2A_1)(F_1F_2 - G_1G_2)} \qquad (2.24)$$

From the above relations it can be observed that for R_2 to be positive, γ must be greater than zero. Hence, decoupling in a laminate is possible only if its material properties and fibre orientations are such that γ is positive. For decoupling,

$$\begin{aligned} \gamma &> 0 \\ R_1 &> 0 \\ R_2 &> 0 \end{aligned} \qquad (2.25)$$

If the fibre orientation in each lamina is same i.e., $\theta_1 = \theta_2 = \theta_3$, expression for γ reduces to

$$\gamma = \left[\frac{B_3}{B_1} - \frac{A_3}{A_1} \right] / \left[\frac{B_2}{B_1} - \frac{A_2}{A_1} \right] \qquad (2.26)$$

For homogeneous laminate,

$$A_1 = A_2 = A_3$$

$$B_1 = B_2 = B_3$$

Hence γ in this case becomes independent of A's and B's or in other words independent of material properties, which is in confirmation with the results obtained by Iyenger [4]. For this case, laminate to be decoupled, thickness ratios must satisfy the relation,

$$R_1 = -(1+R_2) + \sqrt{2[(1+R_2)^2 - 1]} \qquad (2.27)$$

with the conditions,

$$|\theta_1| > |\theta_2|, |\theta_3|$$

or $$|\theta_1| < |\theta_2|, |\theta_3| \qquad (2.28)$$

If the total thickness of the laminate is 2t, then individual lamina thicknesses are given in terms of thickness ratios.

$$\frac{t_1}{t} = \frac{R_1}{1+R_1+R_2}$$

$$\frac{t_2}{t} = \frac{R_2}{1+R_1+R_2} \qquad (2.29)$$

$$\frac{t_3}{t} = 1 - \left(\frac{t_1}{t} + \frac{t_2}{t}\right)$$

2.5 SIMPLIFIED GOVERNING EQUATIONS:

For a completely decoupled antisymmetric laminate since,

$$D_{16} = D_{26} = B_{16} = B_{26} = B_{11} = B_{12} = B_{22} = B_{66} = 0$$

the governing eqn. (2.17) assumes a simpliedf form as,

$$D_{11} w_{,xxxx} + 2(D_{12} + 2D_{66}) W_{,xxyy} + D_{22} w_{,yyyy} = -\lambda (K_1 w_{,xx} + K_2 w_{,yy} + 2K_3 w_{,xy}) \qquad (2.30)$$

Introducing dimensionless quantities,

$$\xi = \frac{X}{a} , \qquad \eta = \frac{Y}{b} \qquad (2.31)$$

where, a and b are the length and width of plate respectively, Eqn. (2.30) takes the form

$$D_{11} w_{,\xi\xi\xi\xi} + 2(D_{12} + D_{66}) R^2 w_{,\xi\xi\eta\eta} + D_{22} R^4 w_{,\eta\eta\eta\eta}$$

$$= -\lambda a^2 (K_1 w_{,\xi\xi} + K_2 R^2 w_{,\eta\eta} + 2K_3 R w_{,\xi\eta}) \quad (2.32)$$

where R is aspect ratio

$$R = \frac{a}{b}$$

2.6 BOUNDARY CONDITIONS:

The governing eqn. (2.32) is solved for following boundary conditions:

(1) <u>All Edges Simply Supported</u>:

Boundary conditions for the plate simply supported on all sides are as given [16]

At $\xi = 0, 1$

$w = 0$

$M_{\xi} = D_{11} w_{,\xi\xi} b^2 + D_{12} w_{,\eta\eta} a^2 = 0$

At $\eta = 0, 1$

$w = 0$

$M_{\eta} = - D_{12} w_{,\xi\xi} b^2 - D_{22} a^2 w_{,\eta\eta}$ (2.33)

eq.(2.32) In the presence of shear terms, due to presence of odd and even derivatives, analysis becomes difficult. Galerkin-Valsov method has been used to solve the equation.

To obtain the solution by Galerkin Valsov method, one needs a displacement function, which satisfies the natural and geometric boundary conditions. For simply supported case it can be expressed as,

$$w = \sum_m \sum_n w_{mn} \phi_{mn} (\xi,\eta) \tag{2.34}$$

where,

$$\phi_{mn} = \sin (m\pi\xi) \sin (n\pi\eta) \tag{2.35}$$

Applying Galerkin's technique to solve eqn. (2.32) with assumed displacement function, one gets

$$\sum_m \sum_n \int_0^1 \int_0^1 [D_{11} \sin m\pi\xi \sin n\pi\eta (m\pi)^4$$

$$+ 2R^2 (D_{12}+2D_{66}) (n\pi)^2(m\pi)^2 \sin m\pi\xi \sin n\pi\eta$$

$$+ D_{22} R^4 \sin m\pi\xi \sin n\pi\eta (n\pi)^4] w_{mn} \sin i\pi\xi \sin j\pi\eta \, d\xi d\eta$$

$$= \sum_m \sum_n - \lambda a^2 \int_0^1 \int_0^1 [-K_1 \sin m\pi\xi \sin n\pi\eta (m\pi)^2$$

$$- K_2 R^2 \sin m\pi\xi \sin n\pi\eta (n\pi)^2 + 2K_3 R \cos m\pi\xi \cos n\pi\eta (m\pi)$$

$$w_{mn} \sin i\pi\xi \sin j\pi\eta \, d\xi \, d\eta \tag{2.36a}$$

$$\sum_m \sum_n \int_0^1 \int_0^1 [D_{11}(m\pi)^4 + 2(D_{12} + 2D_{66}) R^2(n\pi)^2(m\pi)^2$$

$$+ D_{22} R^4 (n\pi)^4] w_{mn} (\sin m\pi\xi \sin i\pi\xi)(\sin n\pi\eta \sin j\pi\eta) d\xi d\eta$$

$$= \sum_m \sum_n a^2\lambda \int_0^1 \int_0^1 [K_1(m\pi)^2 + K_2 R^2(n\pi)^2]$$

$$(\sin m\pi\xi \sin i\pi\xi)(\sin n\pi\eta \sin j\pi\eta) w_{mn} d\xi d\eta$$

$$- \sum_m \sum_n a^2\lambda \int_0^1 \int_0^1 [2K_3 R(m\pi)(n\pi)]$$

$$(\cos m\pi\xi \sin i\pi\xi)(\cos n\pi\eta \sin j\pi\eta) w_{mn} d\xi d\eta \qquad (2.36b)$$

Using the following properties of trigonometric functions:

$$\int_0^1 \sin(m\pi\xi) \sin(i\pi\xi) d\xi = 0 \quad \text{if } m \neq i$$

$$= \frac{1}{2} \quad \text{if } m = i$$

$$\int_0^1 \sin(j\pi\eta) \cos(n\pi\eta) d\eta = 0 \quad \text{if } j = n \text{ or } |n \pm j| \text{ even}$$

$$= \frac{2j}{\pi(j^2 - n^2)} \quad \text{if } j \neq n \text{ or } |n \pm j| \text{ odd}$$

$$\int_0^1 \sin(i\pi\xi) \cos(m\pi\xi) d\xi = 0 \quad \text{if } i = m \text{ or } |m \pm i| \text{ even}$$

$$= \frac{2i}{\pi(i^2 - m^2)} \quad \text{if } i \neq m \text{ or } |m \pm i| \text{ odd}$$

(2.37)

Eqn. (2.36b) reduces to,

$$\sum_m \sum_n [D_{11} (m\pi)^4 + 2(D_{12} + 2D_{66}) R^2 (mn)^2 \pi^4 + R^4 D_{22} (n\pi)^4] w_{mn} = \lambda \sum_m \sum_n w_{mn} [K_1 (m\pi)^2 + K_2 R^2 (n\pi)^2 - 32a^2 K_3 R (mn)(H_{ij})] \quad (2.38)$$

where,

$$H_{ij} = \frac{j}{(j^2-n^2)} \frac{i}{(i^2-m^2)} \quad \text{if } |m\pm i| \text{ odd or } |n\pm j| \text{ odd}$$

$$= 0 \quad \text{if } m = i \text{ or } n = j \text{ or } |m\pm i| \text{ even or } |n\pm j| \text{ even} \quad (2.39)$$

$$\sum_m \sum_n [D_{11} m^4 + 2(D_{12} + 2D_{66}) (mn)^2 R^2 + D_{22} n^4 R^4] w_{mn} = \frac{\lambda . a^2}{\pi^2} \sum_m \sum_n [K_1 m^2 + R^2 K_2 n^2 - 32K_3 mnR (H_{ij})] w_{mn}$$

The general form of eqn. (2.40) can be written in . form

$$\sum_m \sum_n A_{ijmn} \cdot W_{mn} = \lambda \sum_m \sum_n B_{ijmn} \cdot W_{mn}, \quad i = 1,m, \quad j = 1,n \tag{2.41}$$

where,

$$A_{ijmn} = D_{11} \cdot m^4 + 2(D_{12} + 2D_{66}) (mn)^2 R^2 + D_{22} n^4 R^4 \tag{2.42a}$$

$$B_{ijmn} = \frac{a^2}{\pi^2} [K_1 m^2 + R^2 K_2 n^2 - 32 K_3 mn R (H_{ij})] \tag{2.42b}$$

The eqn. (2.41) gives i x j set of homogeneous equations The eqn. (2.41) is eigen-value problem. It can be solved to obtain lowest eigen value, which gives critical buckling load. In matrix form eqn. (2.41) can be written as

$$[A] \{W\} = \lambda [B] \{W\} \tag{2.42c}$$

If shear loading is not present ($\bar{x}_{\xi} = 0$), matrix [A] and matrix [E], in eq. (2.42c), take the form of diagonal matrices.

(2) Clamped- Boundary Conditions:

Boundary conditions for all edges clamped are:

At $\xi = 0, 1$

$$w = 0$$

$$w_{,\xi} = 0 \qquad (2.44a)$$

At $\eta = 0, 1$

$$w = 0$$

$$w_{,\eta} = 0 \qquad (2.44b)$$

For clamped case also, it is not possible to find a closed form solution. Hence as discussed in case of simply supported boundary conditioned plate analysis, in this case also Galerkin's technique can be applied to solve the governing eqn. (2.32). For this case displacement function satisfying the boundary conditions can be expressed as double Fourier Series.

$$w = \sum_m \sum_n w_{mn} X_m(\xi) Y_n(\eta) \qquad (2.45a)$$

where $X_m(\xi)$ and $Y_n(\eta)$ are the corresponding characteristic functions of a clamped vibrating beam. Beam characteristic functions have been found to be fast converging functions. For the present case $X_m(\xi)$ and $Y_n(\eta)$ take the form,

$$X_m(\xi) = \text{Cos h } (\gamma_m \xi) - \text{Cos } (\gamma_m \xi) - C_m [\text{Sin h } (\gamma_m \xi) - \text{Sin } (\gamma_m \xi)] \quad (2.45b)$$

$$Y_n(\eta) = \text{Cosh } (\gamma_n \eta) - \text{Cos } (\gamma_n \eta) - C_n [\text{Sinh } (\gamma_n \eta) - \text{Sin } (\gamma_n \eta)] \quad (2.45c)$$

where γ_m and γ_n are the roots of the characteristic equation,

$$\text{Cosh } \gamma_i \text{ Cos } \gamma_i = 1 \quad (2.45d)$$

and the parameter C_i s are obtained from relation,

$$C_i = (\text{Cosh } \gamma_i - \text{Cos } \gamma_i)/(\text{Sinh } \gamma_i - \text{Sin } \gamma_i) \quad (2.45e)$$

Now substituting eqns. (2.42) in governing eqn. (2.32) and applying Galerkin's technique, one gets,

$$\sum_m \sum_n \int_0^1 \int_0^1 [w_{mn} D_{11} X_{m,\xi\xi\xi\xi} Y_n + 2(D_{12}+2D_{66}) X_{m,\xi\xi} Y_{n,\eta\eta} R^2 + D_{22} X_m Y_{n,\eta\eta\eta\eta} R^4] X_i Y_j \, d\xi \, d\eta$$
$$= - \sum_m \sum_n a^2 \lambda \int_0^1 \int_0^1 [K_1 X_{m,\xi\xi} Y_n + 2K_3 X_{m,\xi} Y_{n,\eta} R + K_2 R^2 X_m Y_{n,\eta\eta}] w_{mn} X_i Y_j \, d\xi \, d\eta \quad (2.46)$$

The following orthogonal properties of beam characteristic functions result in further simplification of above eqn.

$$\int_0^1 X_{m,\xi\xi\xi\xi}(\xi) X_n(\xi) \, d\xi = 0 \quad \text{if } m \neq n$$
$$\neq 0 \quad \text{if } m = n$$
$$\int_0^1 X_m(\xi) X_n(\xi) \, d\xi = 0 \quad \text{if } m \neq n$$
$$\neq 0 \quad \text{if } m = n \quad (2.47)$$

Now defining the following integrals,

$$M_i^m = \frac{\int_0^1 X_{m,\xi}(\xi)\, X_i(\xi)\, d\xi}{\int_0^1 X_i^2(\xi)\, d\xi}$$

$$L_i^m = \frac{\int_0^1 Y_m(\eta)\, Y_j(\eta)\, d\eta}{\int_0^1 Y_j^2(\eta)\, d\eta}$$

$$L_j^n = \frac{\int_0^1 Y_{n,\eta\eta}(\eta)\, Y_j(\eta)\, d\eta}{\int_0^1 Y_j^2(\eta)\, d\eta}$$

$$J_i^m = \frac{\int_0^1 X_{m,\xi\xi\xi\xi}(\xi)\, X_i(\xi)\, d\xi}{\int_0^1 X_i^2(\xi)\, d}$$

$$J_j^n = \frac{\int_0^1 Y_{n,\eta\eta\eta\eta}(\eta)\, Y_i(\eta)\, d\eta}{\int_0^1 Y_j^2(\eta)\, d\eta} \qquad (2.48)$$

Equation (2.46) reduces to,

$$\sum_m \sum_n \left[D_{11} J_i^m \delta_{nj} + 2(D_{12}+2D_{66}) L_i^m L_j^n R^2 \right.$$

$$\left. + R^4 D_{22} J_j^n \delta_{mi} \right] W_{mn}$$

$$= - \sum_m \sum_n \lambda a^2 \left[K_1 L_i^m \delta_{nj} + 2K_3 K_j^n M_i^m R \right.$$

$$\left. + K_2 R^2 L_j^n \delta_{mi} \right] W_{mn} \qquad (2.49)$$

where, $\delta_{ij} = 0$ for $i \neq j$

$\delta_{ij} = 1$ for $i = j$

The general form of the eqn. (2.49) can be written in the form

$$\sum_m \sum_n A_{ijmn} W_{mn} = \lambda \sum_m \sum_n B_{ijmn} W_{mn}, \quad \begin{matrix} i = 1,m \\ j = 1,n \end{matrix}$$

(2.50)

where,

$$A_{ijmn} = D_{11} J_i^m \delta_{nj} + 2 (D_{12}+2D_{66}) L_i^m L_j^n R^2$$

$$+ R^4 D_{22} J_j^n \delta_{mi} \qquad (2.51a)$$

$$B_{ijmn} = -a^2 [K_1 L_i^m n_j + 2K_3 M_j^n M_i^m R + K_2 R^2 L_j^n m_i] \quad (2.51b)$$

The eqn. (2.50) is a eigen value problem, and it can be solved to get lowest eigen value, which gives buckling load.

(3) Clamped and Simply Supported Rectangular Plate Analysis:

Boundary conditions, when the plate is simply supported at two edges parallel to x-axis and clamped along the other two edges are,

$$\begin{aligned} \text{At } \xi &= 0, 1 \\ w &= 0 \\ M_\xi &= 0 \\ &= D_{11} w_{,\xi\xi} b^2 + D_{12} w_{,\eta\eta} a^2 \\ \text{At } \eta &= 0, 1 \\ w &= 0 \\ w_{,\eta} &= 0 \end{aligned} \quad (2.52)$$

In this case all the equations are same as derived in the case of clamped boundary conditions, with the only difference that characteristic eqn. (2.45d) is replaced by,

$$\tan \lambda = \tanh \lambda \quad (2.53)$$

2.7 METHOD OF OPTIMIZATION:

Since the objective function for composite laminated plate is quite a complicated function of design variables, fibre orientations and volume functions, constrained optimization method without using derivatives is suitable for solving the problem.

For the constrained optimization solution without using derivatives, sequential unconstrained minimization method (SUMT), Exterior Penalty Function method has been used.

In the exterior penalty function method, objective function - ϕ is generally taken as,

$$\phi(X, r_k) = f(X) + r_k \sum_{j=1}^{m} \langle g_j(X) \rangle^q \qquad (2.54)$$

where r_k is a positive penalty parameter, the exponent q is a non-negative constant, and the bracket function $\langle g_j(X) \rangle$ is defined as,

$$\langle g_j(X) \rangle = \max \langle g_j(X), 0 \rangle = \begin{cases} g_j(X) \text{ if } g_j(X) > 0 & \text{(constraint is violated)} \\ 0 \text{ , if } g_j(X) \leq 0 & \text{(constraint is satisfied)} \end{cases} \qquad (2.55)$$

Usually, the function $\phi(X, r_k)$ possesses a minimum as a function of X in the infeasible region. The unconstrained minima X_k^* converge to the optimal solution of the original problem as $K \to \infty$ and $r_k \to \infty$. Thus the unconstrained minima

approach the feasible domain gradually as $K \to \infty$, the X_k eventually lies in the feasible region.

For unconstrained minimization, Powell's conjugate direction method has been used. Powell's method is an extension of the basic pattern search method. It is the most widely accepted direct search methods and is a method of conjugate directions. Powell's method is a very powerful method and has been proved to be superior to most of the descent methods.

In the present optimization study, a self-developed programme has been used. The programme was extensively checked for various optimization problems.

2.8 STEPS IN PROGRAMMING:

The main steps of programming procedure to determine the lowest eigen value (critical buckling load) are as follows:

- For given fibre and matrix properties, E_1, E_2, G_{12}, ν_{12}, $[Q]$ and $[\bar{Q}]$ are generated.
- With the quantities calculated above, the thickness ratios (R_1 and R_2) and percentage thicknesses of the layers are calculated to make the laminate decoupled.
- For specified geometry and thickness, coefficients of stiffness matrices D_{ij} are computed.
- Using NAG Subroutine FO2BJF available with NAG Library of DEC 1090 System, the lowest eigen value is calculated.
- The procedure is repeated till the values of design variables i.e. fibre orientations and fibre volume fractions are found, which make laminate decoupled and give maximum buckling load.

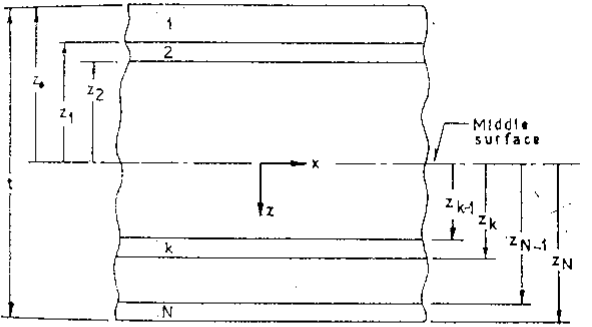


FIG. 2·1: GEOMETRY OF A n – LAYERED LAMINATE.

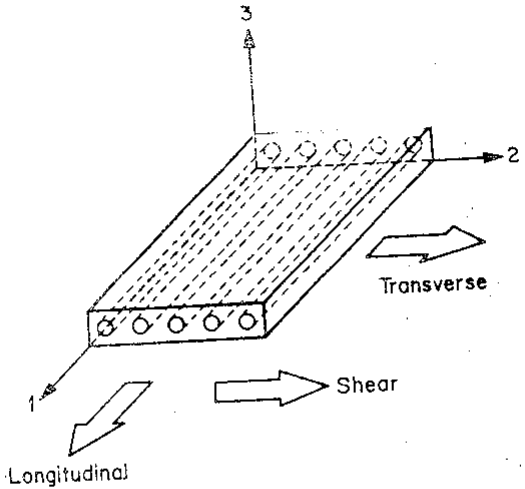


FIG 2.2 : UNIDIRECTIONALLY REINFORCED LAMINA

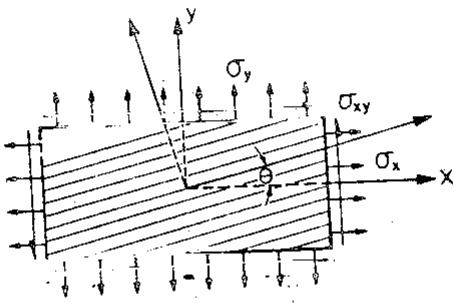


FIG. 2.3 : POSITIVE ROTATION OF LAMINA AXES BY θ° W.R.T. LAMINATE AXES.

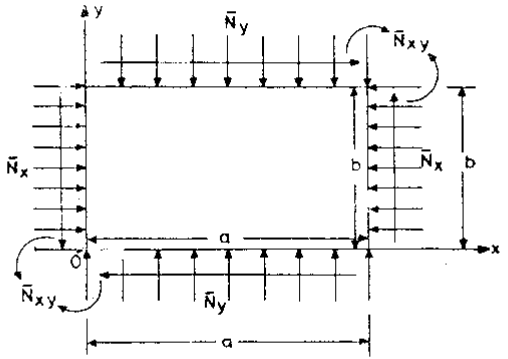
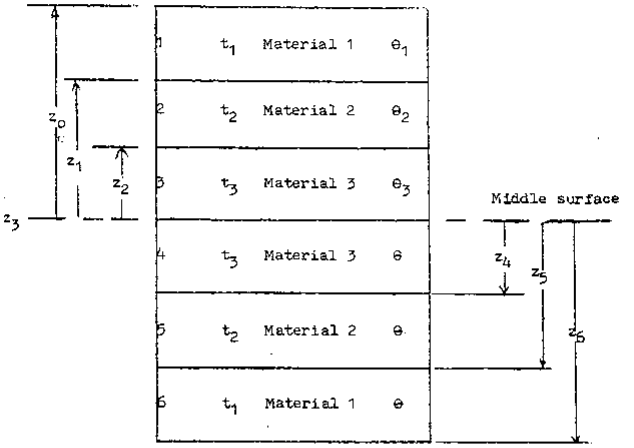


FIG. 2.4: LOADING AND GEOMETRY OF RECTANGULAR PLATE.



$$z_0 = -z_6 = -(t_1 + t_2 + t_3)$$

$$z_1 = -z_5 = -(t_2 + t_3)$$

$$z_2 = -z_4 = -t_3$$

$$z_3 = 0$$

Fig.2.5 Geometry of a six-layered hybrid laminate.

CHAPTER III

OPTIMIZATION OF HYBRID LAMINATED PLATE - I

Generally, hybrid laminates refer to the laminates made of more than one type of fibres. However, laminates made of one type of fibres, but different fibre packing in each layer, also behave as hybrids. In this chapter, an attempt is made to optimize, such kind of hybrid laminated plate, subjected to a system of in plane forces. The study is made for (1) simply supported, (2) clamped and (3) two sides clamped and two simply supported, boundary conditions.

3.1 OPTIMIZATION:

The objective function of the present optimization study is to maximize the buckling strength of the laminated plate of specified total thickness and total fibre content or total weight, as fibres of only one material are used. Design variables are fibre orientations and fibre volume fractions in each ply. Investigations are carried out on six layered anti-symmetric angle ply laminate for different loading combinations and aspect ratios. The objective function is defined as,

$$\phi = \frac{a^2 \lambda_{cr}}{t^3}$$

3.2 PROBLEM FORMULATION:

Maximize $(\phi\ (\theta_i,\ V_{fi})$

or

Minimize $(-\phi\ (\theta_i,\ V_{fi})$

$i = 1, 3$

$$\phi = \frac{a^2 - \lambda_{cr}}{t^3}$$

subject to the following constraints:

(1) Lower and upper bounds on fibre orientations

$$0^\circ \leq |\theta_i| \leq 90^\circ$$

(2) Lower bounds on individual ply thickness

$$t_{min} \leq t_i$$

For numerical computations, t_{min} is taken as 6 percent of total thickness on the basis of practical considerations, of the minimum thickness available.

(3) Equality constraints on the total fibre content.

$$2 \sum_{i=1}^{3} V_{fi}\ t_i = V_f \text{ (specified)}$$

where V_{fi} is the fibre volume fraction in i-th layer. V_f (specified), is the specified total fibre content. In present case, V_f (specified) = 0.6, which is equivalent to the total fibre in a homogeneous laminate with 60 percent volume fraction.

(4) Lower and upper bounds on fibre volume fractions in each lamina

$$V_{f\,min} \leq V_{fi} \leq V_{f\,max}$$

Keeping in view, the practical limits on fibre packing, the following values are used:

$$V_{f\,min} = 10 \text{ percent}$$

$$V_{f\,max} = 75 \text{ percent}$$

(5) Constraints on γ to ensure decoupling

$$\gamma > 0$$

(6) Constraints on R_1 and R_2, to have positive thicknesses of the laminae

$$R_1 > 0$$

$$R_2 > 0$$

3.3 NUMERICAL COMPUTATIONS:

Numerical computations were carried out on DEC-1090 system, available at IIT Kanpur.

In order to develop confidence and to check the present formulation, the computer programme was extensively checked. In this process, results were obtained on six layered antisymmetric boron-epoxy homogeneous laminates and compared with already existing results of Barway [13]. For the sake of comparison, the objective function and the material properties are same as that of Barway [13] and are listed below for completeness.

Property	Boron-Epoxy Composite Material
$E_1 \times 10^{-6}$ (kg/cm^2)	2.11
$E_2 \times 10^{-5}$ (kg/cm^2)	2.11
$G_{12} \times 10^{-4}$ (kg/cm^2)	7.03
ν_{12}	0.30

The objective function ϕ is non dimensional buckling load and defined as,

$$\phi = \frac{12\, b^2 \lambda_{cr}}{\pi^2 t^3 (Q_{22})_{Boron}}$$

Sequence of numerical computations in the present work is as follows.

Computation were carried out on boron-epoxy composite system. The fibre and matrix properties used [14] are as given below:

Property	Value
E_f	3.97×10^6 (kg/cm^2)
E_m	3.38×10^4 (kg/cm^2)
ν_f	0.2
ν_m	0.2

The aim of the present work is to show that considerable load carrying capacity of the laminates can be increased, by allocating different fibre volume fractions to the laminae of a hybrid laminate. To justify the values used, effect of ν_f and ν_m has been shown on percentage difference in loads of hybrid and homogeneous laminates. For achieving required accuracy in computing the results, convergence study was made. Starting point effect was also studied, before computing final results. Initially, fibre orientations in each layer were taken as separate variables. However, almost same results were obtained by taking same orientations in all the layers. By making use of above observations, finally results were obtained and compared for hybrid and homogeneous laminates, for various loading combinations, aspect ratio and boundary conditions.

3.4 RESULTS AND DISCUSSION:

As already mentioned in the process of checking of present work, certain results were obtained, for the sake of comparison. Results were compared with that of Barway [13], for six layered antisymmetric homogeneous laminates, for simply-supported and clamped boundary conditions. Following loading combinations have been considered:

(i) Uniaxial compressive load ($K_2 = K_3 = 0$)

(ii) Biaxial compressive load ($K_3 = 0$)

(iii) Shear load ($K_1 = K_2 = 0$)

(iv) Combined shear and compressive loads ($K_2 = 0$)

Barway [13] reported, that at optimum point fibres in all the layers tend to align in same direction. So, for comparison, fibre orientations in all laminae are taken to be same. Comparison is made in Table 1. It is observed that in all the cases, the non-dimensional buckling loads obtained in present work are either same or slightly greater than that obtained by [13]. The reason is that, [13] has solved three coupled governing equations. While, in present work, as decoupling is ensured, at each step of optimization, it is required to solve only one decoupled governing equation. Here analysis becomes much simpler, so considerable computation time is saved. This comparison proves the authenticity of the present work.

In the present work, composite properties are generated from fibre and matrix properties with the help of empirical relations. So it becomes necessary to justify the values of ν_f and ν_m used. As one is interested in the percentage difference in loads between hybrid and homogeneous laminates, effect of ν_f and ν_m is studied on the percentage difference. Table 2 shows that there is no considerable difference (.3 percent) in percentage differences in loads for various combinations of ν_f and ν_m. Hence, any values of ν_f and ν_m in practical range can be taken without affecting final results.

In order to achieve, the desired accuracy in computing of buckling load, convergence study is made. It was found that, in the presence of shear load ($K_3 \neq 0$), eigen value (critical buckling load) converges upto four decimal places, with $m = n = 7$

in eqns. (2.40) and (2.49). In an attempt to reduce computation efforts, it was observed that considerable time is saved without much loss of accuracy by running optimization programme with $m = n = 4$, instead of $m = n = 7$ and then using the design variables so obtained to compute eigen value with $m = n = 7$. Results are shown in Table 3. However, in the absence of shear load ($K_3 = 0$), eigen value becomes independent of m and n chosen, in case of simply supported and results can be obtained with $m = n = 1$. The reason is that in this case exact solution of eqn. (2.43) can be obtained.

During computations, it was observed that optimization results are sensitive to volume fraction starting point. But no sensitivity was found to fibre orientation starting point. Table 4 shows the starting point study for simply supported, unioxial loading case. It was observed that starting with highest volume fractions in outer layer and minimum in inner layer, yields best results. However, there is no appreciable difference in buckling loads with different starting points.

Barway [13] reported that at optimum point, fibre angles in all the layers are almost same. Table 5 compares the results between the two cases, (1) when angles in each layer is taken as independent variable and (2) when angles in all layers are same and taken as one variable only. It is observed that there is negligible difference (0.02 percent) in loads between the two cases. Hence in further computations, fibre orientations in all the layers are taken same.

Table 5 shows the detailed optimization results for hybrid and homogeneous laminates for simply supported boundary conditions. It shows for various loading combinations and aspect ratios, optimum volume fractions and fibre orientation for homogeneous and hybrid laminates the percentage thicknesses of individual ply of hybrid laminate, which ensure decoupling are also shown. However, percentage thicknesses in case of homogeneous laminates are not shown. As observed from decoupling analysis, in case of homogeneous laminates thickness ratios R_1 and R_2 are related by the relation, eqn. 2.27,

$$R_1 = -(1 + R_2) + \sqrt{2\,[(1+R_2)^2 - 1]}$$

Thus any combinations of R_1 and R_2, which satisfy the above relation, decouple the laminate. Table 7, shows that homogeneous laminates do not have unique thicknesses, and different combinations of R_1 and R_2 give different thicknesses for the same load.

Study of Table 6 shows that optimum fibre volume fractions are almost same irrespective of aspect ratios and loading combinations. In all the cases, almost maximum possible fibre fraction is assigned to outermost layers, and slightly less to inner layers and minimum to innermost layers. In the present problem, the upper and lower limits on volume fractions are 75 percent and 10 percent respectively. In most of the cases, optimal volume fractions in the layers are 74.9 percent, 74.5 percent and 10.2 percent. Thus it is observed that if fibre fractions are allowed to vary from layer to layer, with total fibre content

fixed, they should be allocated such that maximum goes to outer layer, slightly less to inner and minimum to innermost layers. It is also observed from table 5, that percentage thicknesses of the layers are also not sensitive to aspect ratios and loading combinations. In most of the cases, 13.5 percent of total thickness is contributed by each outer most layer, 25.5 percent second layer and 11 percent by the third layer.

Table 7 also shows the buckling loads for hybrid and homogeneous laminates. Total fibre content in both the cases is same. It is observed, that hybrid laminates have at least 25 percent higher load carrying capacity than homogeneous laminates. Since total fibre content and fibre material are same in both the cases, it can be inferred that for the same cost and weight, hybrid laminates can give as high as 25 percent extra buckling strength than homogeneous laminates.

Table 8 shows, the optimization results for clamped boundary conditions. Results for the two sides simply supported and the two clamped, boundary conditions are shown in Table 9. It is observed that distribution of thicknesses, and volume fractions is same as obtained in the case of simply-supported boundary conditions. In these cases also hybrid laminates have as high as 24.5 percent higher buckling strength than homogeneous laminates.

It is observed in Table 6, Table 8 and Table 9 that optimum volume fractions and thicknesses are almost same for all loading combinations, aspect ratios and boundary conditions. So a

study is made, in which volume fractions and thicknesses were specified, as obtained from above tables and only fibre orientation was used as a design parameter to optimize the buckling load.

Following values were used,

$V_{f1} = 0.75$, $V_{f2} = 0.74$, $V_{f3} = 0.102$

$t_1 = 13.6$ percent, $t_2 = 25.5$ percent

and $t_3 = 10.9$ percent.

The results obtained for simply supported case are shown in Table 10. Since now, the laminate is not exactly decoupled, B_{16} and B_{26} elements of coupling matrix were computed and compared with minimum of non-zero D_{ij} s. Results show that B_{16} and B_{26} are of the order of 10^{-4} as compared to elements of [D] matrix, so the eqn. used for assuming decoupled laminate is approximately valid. It is observed that buckling loads in this case are slightly reduced (2 ~ 3 percent) than that previously obtained.

Figs. 2.1, 2.2 and 2.3. show the buckling load vs. aspect ratio curves for (1) simply supported, (2) clamped and (3) two sides clamped and two simply supported boundary conditions respectively. For each boundary conditions, curves are drawn for various combinations of loadings. It is seen that buckling load (which is defined as $\phi = a^2 \lambda_{cr}/t^3$), increases with increase in aspect ratio. Buckling load increases slowly for aspect ratio from 0.5 to 1, but increases much faster, with

aspect ratio more than 1. The reason is, the presence of second and fourth powers of aspect ratio in eqns. (2.41) and (2.49).

3.5 CONCLUSIONS:

1. If decoupling is ensured, the analysis of laminate under inplane loads, becomes much simpler. This gives accurate results with considerable saving in computation time.

2. Fiber and matrix properties like ν_f and ν_m, have insignificant effect on the percentage difference between loads of hybrid and homogeneous laminate.

3. At optimum point fibres in all layers of a laminate tend to orient in the same direction. So considerable computation time can be saved in optimization process. by a taking orientations in each layer same and as one variable.

4. Optimum fibre volume fractions and also percentage thicknesses of the plies, are insensitive to aspect ratio, loading combinations and boundary conditions.

5. Optimum fibre orientations, depend on aspect ratios, loading combinations and boundary conditions.

6. For buckling strength of a laminate to be maximum, volume fraction, in the outermost layer (V_{f1}) must be maximum possible, slightly less in second layer (V_{f2}) and minimum in innermost layer (V_{f3}) and thicknesses must be t_1 = 13.6 percent, t_2 = 25.5 percent and t_3 = 10.9 percent.

With these values prescribed only fibre orientation can be chosen as design variable in optimization study without much loss (2 - 5 percent) in buckling strength.

7. At optimum point, there exists unique thickness ratios, of the layers which ensure decoupling, in case of hybrid laminates. However, in case of homogeneous laminates, infinite combinations of thickness ratios (R_1 and R_2) are possible.

8. Hybrid laminates, made of different fibre packings in each layer, give as high as 25 percent, higher buckling strength, than homogeneous laminates with fixed fibre volume fractions, for the same cost and weight.

Six layered homogeneous Laminate

Non Dimensional Load = $12b^2 \lambda_{cr}/(\pi^2 t^3)$

K_1	K_2	K_3	AR	BARWEY [13] RESULTS							PRESENT RESULTS				
				θ_1	θ_2	θ_3	% Thickness t_1	% Thickness t_2	% Thickness t_3	Non Dim. Load	θ	% Thickness t_1	% Thickness t_2	% Thickness t_3	Non Dim. Load
								SIMPLY SUPPORTED							
1	0	0	1	45	-45	45	10.9	22.4	16.7	22.0	45	9.7	21.0	19.3	22.3
			2	45	-45	45	10.9	22.4	16.7	22.0	45	10.8	22.4	16.8	22.3
1	1	0	1	45	-45	45	10.9	22.4	16.7	11.0	45	10.8	22.4	16.8	11.75
			2	71.98	-71.55	-71.58	10.9	22.4	16.7	8.056	71.97	8.9	20.1	21.0	8.084
0	0	1	1	45	-45	45	10.9	22.4	16.7	50.125	45	9.7	21.0	19.3	50.76
			2	57.22	-57.21	57.96	10.9	22.4	16.7	34.75	57.51	10.7	22.0	17.3	34.99
1	0	1	1	45	-45	45	10.9	22.4	16.7	19.039	44.8	8.8	20.0	21.2	19.15
			2	48.96	-48.97	48.0	10.9	22.4	16.7	16.661	48.64	7.8	19.2	23.0	16.80
								CLAMPED							
1	0	0	1	0	0	0	16.66	16.66	16.66	46.29	0	8.0	19.3	22.7	46.3
			2	43.69	-43.72	43.70	16.66	16.66	16.66	36.99	43.68	7.8	19.2	22.9	37.0
1	1	0	1	32.99	-32.0	32.99	10.9	22.4	16.7	22.98	33.71	10.2	21.5	18.3	23.1
			2	61.1	-60.9	60.08	10.9	22.4	16.7	21.519	61.0	9.6	20.8	19.6	21.66

Simply Supported
Uniaxial Loading
AR = 1.5

γ_f	γ_m	BUCKLING LOAD ϕ (Kg/cm^2) x 10^5		Percent Difference
		Hybrid (1)	Homogeneous (2)	$\frac{(1)-(2)}{(2)}$ x 100
0.2	0.2	21.956	17.212	27.56
0.3	0.25	22.08	17.30	27.68
0.3	0.3	22.104	17.305	27.74
0.35	0.3	22.139	17.31	27.9

Table 3.3: Convergence Study.

Simply Supported ϕ is Buckling load $(kg/cm^2) \times 10^{-6}$

AR	θ	Optimization with 4 Terms							Optimization with 7 Terms							Per cer Dif ren
					% Thickness								% Thickness			
		V_{f1}	V_{f2}	V_{f3}	t_1	t_2	t_3	ϕ	θ	V_{f1}	V_{f2}	V_{f3}	t_1	t_2	t_3	ϕ	ϕ
		$K_1 = 0,\ K_2 = 0,\ K_3 = 1$															
0.5	32.37	.749	.744	.124	13.5	25.4	10.9	8.53	32.41	.749	.707	.462	9.26	19.18	21.55	8.55	.
2.0	57.6	.749	.745	.137	13.5	25.1	11.2	34.14	57.5	.749	.707	.462	9.26	19.18	21.55	34.14	-
		$K_1 = 1,\ K_2 = 0,\ K_3 = 1$															
0.5	0	.749	.711	.124	13.5	25.4	10.9	2.411	0	.749	.707	.462	9.26	19.18	21.55	2.413	.1
2.0	48.6	.749	.745	.137	13.5	25.1	11.2	16.51	48.59	.749	.707	.462	9.26	19.18	21.55	16.612	.

Table 3.4: Study of Starting Point Effect.

Simply Supported

Uniaxial Loading

AR = 1 ϕ is Buckling Load $(kg/cm^2)x10^{-6}$

STARTING POINT				OPTIMUM POINT							
θ	V_{f1}	V_{f2}	V_{f3}	ϕ	θ	V_{f1}	V_{f2}	V_{f3}	percent Thickness t_1	t_2	t_3
45	0.6	0.6	0.6	5.004	45	0.69	0.687	0.164	14.1	28.4	7.5
42	0.75	0.7	0.1	5.4637	45	0.75	0.742	0.2	13.2	24.3	12.3
45	0.7	0.6	0.4	5.4663	45	0.749	0.742	0.175	13.3	24.7	11.9
45	0.749	0.745	0.1	5.4839	45	0.749	0.745	0.137	13.5	25.2	11.2
40	0.749	0.745	0.1	5.4839	45	0.749	0.745	0.137	13.5	25.2	11.2

Table 3.5: Study of Optimum Fibre Orientations.

Simply Supported

AR = 2 ϕ is Buckling Load (kg/cm^2)$\times 10^{-6}$

$\theta_1 = \theta_2 = \theta_3$								θ_1, θ_2 and θ_3 As Independent Variables									Diff. in ϕ
θ	V_{f1}	V_{f2}	V_{f3}	% Thickness t_1	t_2	t_3	ϕ	θ_1	θ_2	θ_3	V_{f1}	V_{f2}	V_{f3}	% Thickness t_1	t_2	t_3	ϕ
								$K_1 = 1, K_2 = 0, K_3 = 0$									
45	.749	.745	.107	13.6	25.5	10.7	21.956	45.07	44.95	44.99	.749	.745	.107	13.7	25.5	10.7	21.99
								$K_1 = 1, K_2 = 1, K_3 = 0$									
72	.748	.745	.112	13.6	25.5	10.8	7.9	71.96	71.85	71.96	.748	.745	.112	13.8	25.4	10.7	7.91
								$K_1 = 0, K_2 = 0, K_3 = 1$									
57.62	.749	.745	.137	13.5	25.1	11.2	34.14	57.65	56.51	57.63	.749	.745	.138	13.4	25.2	11.4	34.14
								$K_1 = 1, K_2 = 0, K_3 = 1$									
48.62	.749	.745	.137	13.5	25.1	11.2	16.51	49.0	48.5	49.0	.749	.745	.137	13.7	25.03	11	16.54

Table 3.6: Optimization Results for Simply Supported Plate

Six Layered Antisymmetric Laminate

K_1	K_2	K_3	AR	HYBRID								Homogeneous		Percent Difference $\frac{1-2}{2} \times 100$
				Volume Fractions			% Thickness							
				V_{f1}	V_{f2}	V_{f3}	t_1	t_2	t_3	θ	Buck$^+$ ①	θ	Buck$^+$ ②	
			0.5	0.749	0.745	0.121	13.6	25.4	11.0	0	2.57	0	2.05	26.4
			0.75	0.749	0.745	0.141	13.5	25.2	11.3	36	3.33	35.75	2.63	26.7
1	0	0	1.0	0.749	0.745	0.137	13.5	25.2	11.3	45	5.48	45.0	4.30	24.9
			1.5	0.749	0.745	0.114	13.4	25.0	11.5	42.6	13.14	42.8	10.31	27.5
			2.0	0.749	0.745	0.107	13.6	25.5	10.9	45.0	21.95	45.0	17.21	27.5
			0.5	0.749	0.745	0.136	13.5	25.2	11.3	18.0	1.97	18.6	1.54	27.9
			0.75	0.749	0.745	0.140	13.5	25.1	11.4	36.0	2.15	35.75	1.68	27.0
1	1	0	1.0	0.749	0.745	0.124	13.6	25.4	11.0	45.0	2.74	45.0	2.15	27.4
			1.5	0.749	0.745	0.104	13.6	25.6	10.8	60.2	4.52	60.7	3.56	26.9
			2.0	0.748	0.745	0.112	13.6	25.6	10.8	72.0	7.9	70.7	6.24	26.6
			0.5	0.749	0.745	0.124	13.6	25.4	10.9	32.3	8.53	32.4	6.73	26.7
			0.75	0.748	0.745	0.103	13.6	25.6	10.6	38.8	9.82	38.9	7.75	26.7
0	0	1	1.0	0.749	0.745	0.113	13.6	25.4	10.8	45.0	12.37	45.0	9.77	26.6
			1.5	0.749	0.745	0.137	13.5	25.1	11.2	53.8	20.82	53.0	16.43	26.7
			2.0	0.749	0.745	0.137	13.5	25.1	11.2	57.6	34.14	57.5	26.9	26.9
			0.5	0.749	0.742	0.116	13.5	25.2	11.2	1.6	2.41	0	1.92	25.5
			0.75	0.749	0.745	0.112	13.5	25.2	11.2	36.0	3.05	36.0	2.40	27.0
1	0	1	1.0	0.749	0.745	0.110	13.6	25.5	10.8	45.0	4.72	45.0	3.70	27.5
			1.5	0.749	0.745	0.110	13.6	25.5	10.8	46.0	10.10	46.0	7.97	26.7
			2.0	0.749	0.745	0.137	13.5	25.3	11.2	48.6	16.51	48.7	12.99	27.1

Buck$^+$ = Buckling load ϕ (kg/cm^2) x 10^{-6}

Table 3.7: Optimum Thickness Ratios for Homogeneous Laminate.

Simply Supported

Uniaxial Loading

AR	θ	R_1	R_2	Percent Thickness t_1	t_2	t_3	Optimum Load ϕ (kg/cm^2) x 10^{-6}
0.75	35.75	1.882	3.896	13.8	28.7	7.3	2.6348
	35.75	0.321	0.806	10.71	22.15	17.1	2.6348
	35.75	0.625	1.293	7.8	19.2	23.0	2.6348
1.00	45.0	1.612	3.296	13.6	27.9	8.4	5.4805
	45.0	2.218	4.661	14.1	29.5	6.3	5.4805
	45.0	0.625	1.293	10.7	22.1	17.1	5.4805

Table 3.8: Optimization Results for Clamped Plate.
Six Layered Antisymmetric Laminate ϕ is Buckling Load $(kg/cm^2)x10^{-6}$

K_1	K_2	K_3	AR	HYBRID V_{f1}	V_{f2}	V_{f3}	% Thickness t_1	t_2	t_3	θ	ϕ_1	Homogeneous θ	ϕ_2	Percent Difference in ϕ $\frac{1-2}{2}$ x100
1	0	0	0.5	.749	.746	.100	13.7	25.6	10.7	0	10.827	0	8.062	24.35
			0.75	.749	.745	.114	13.6	25.5	10.9	0	10.421	0	8.303	25.5
			1.0	.749	.745	.114	13.6	25.5	10.9	0.2	11.316	0.02	8.926	28.2
			1.5	.749	.745	.114	13.6	25.5	10.9	33.8	22.007	34.67	17.13	27.5
			2.0	.749	.745	.114	13.6	25.5	10.9	43.68	36.485	43.78	28.401	28.46
1	1	0	0.5	.749	.746	.138	13.6	25.2	11.2	29.0	5.299	29.9	4.139	28.01
			0.75	.749	.746	.137	13.5	25.2	11.3	28.2	5.548	29.36	4.313	28.60
			1.0	.749	.745	.138	13.6	25.2	11.2	56.5	5.657	56.45	4.092	28.83
			1.5	.749	.745	.114	13.6	25.5	10.9	60.16	12.315	59.34	9.589	28.43
			2.0	.749	.745	.116	13.7	25.5	10.8	60.8	21.192	60.07	16.559	27.99
0	0	1	0.5	.749	.745	.138	13.6	25.2	11.2	25.36	13.317	25.85	10.440	27.55
			0.75	.749	.745	.138	13.6	25.2	11.2	32.7	14.653	33.2	11.439	28.1
			1.0	.749	.745	.137	13.5	25.2	11.3	45.02	17.089	44.98	13.314	28.34
			1.5	.749	.745	.138	13.5	25.2	11.3	54.9	31.738	54.5	24.777	28.09
			2.0	.749	.745	.138	13.5	25.2	11.3	64.6	51.921	64.0	41.762	24.32
1	0	1	0.5	.749	.745	.114	13.6	25.5	10.9	16.15	7.031	17.35	5.531	27.1
			0.75	.749	.745	.116	13.6	25.5	10.9	14.13	7.386	16.88	5.791	27.55
			1.00	.749	.745	.115	13.6	25.2	11.2	36.8	8.102	36.7	6.301	28.58
			1.5	.749	.745	.115	13.6	25.2	11.2	44.18	15.473	44.0	12.045	28.45
			2.0	.749	.745	.116	13.6	25.2	11.2	51.0	26.344	50.6	20.375	28.31

Table 3.9: Optimization Results for Simply Supported and Clamped Plate.

Six Layered Antisymmetric Laminate

ϕ is Buckling load (kg/cm^2) x 10^{-6}

K_1	K_2	K_3	AR	HYBRID								HOMOGENEOUS		Percent Difference in ϕ
				V_{f_1}	V_{f_2}	V_{f_3}	% Thickness t_1	t_2	t_3	θ	ϕ_1	θ	ϕ_2	$\frac{\phi_1-\phi_2}{\phi_2} \times 100$
1	0	0	0.5	0.749	.745	.114	13.6	25.5	10.9	0	5.16	0	4.14	24.5
			0.75	0.749	.745	.114	13.6	25.5	10.9	0.2	5.44	0.1	4.31	26.2
			1.0	0.749	.745	.116	13.7	25.5	10.8	42.8	7.49	42.9	5.85	28.1
			1.5	0.749	.745	.116	13.7	25.5	10.8	41.8	16.3	42.05	12.74	27.9
			2.0	0.749	.745	.114	13.7	25.5	10.8	44.8	28.18	44.8	22.03	27.8
1	1	0	0.5	0.749	.745	.116	13.6	25.5	10.9	24.7	3.43	25.4	2.70	27.0
			0.75	0.749	.745	.116	13.7	25.5	10.8	24.25	3.45	25.01	2.72	27.1
			1.0	0.749	.745	.116	13.7	25.5	10.8	49.9	3.91	44.98	3.05	28.1
			1.5	0.749	.745	.114	13.6	25.5	10.9	60.2	7.777	65.2	6.11	27.1
			2.0	0.749	.745	.115	13.7	25.5	10.8	68.3	13.75	64.5	10. 83	26.8
0	0	1	0.5	0.749	.745	.116	13.7	25.5	10.8	31.35	10.67	31.5	8.37	27.4
			0.75	0.749	.745	.114	13.6	25.5	10.9	34.7	12.03	34.95	9.41	27.7
			1.0	0.749	.745	.116	13.7	25.5	10.8	45.0	14.53	45.0	11.35	27.9
			1.5	0.749	.745	.116	13.7	25.5	10.8	57.3	26.1	56.9	20.42	27.7
			2.0	0.749	.745	.114	13.6	25.5	10.8	58.6	42.69	58.48	33.49	27.4
1	0	1	0.5	0.749	.745	.114	13.7	25.5	10.8	1.25	4.55	12.25	3.60	26.5
			0.75	0.749	.745	.115	13.7	25.5	10.8	1.75	4.62	15.1	3.65	26.7
			1.0	0.749	.745	.115	13.7	25.5	10.8	42.88	6.06	43.0	4.74	28.1
			1.5	0.749	.745	.116	13.7	25.5	10.8	46.73	12.37	46.7	9.66	28.1
			2.0	0.749	.745	.114	13.7	25.5	10.8	50.3	20.89	50.18	16.34	27.8

Table 3.10: Comparison of Buckling Load in Exactly Decoupled and Approximately Decoupled Laminate.

Simply Supported

K_1	K_2	K_3	AR	ϕ_{opt} Decoupled (kg/cm^2) $\times 10^{-6}$ 1	ϕ_{opt} Coupled (Kg/cm^2) $\times 10^{-6}$ 2	Percent Difference $\frac{1-2}{1}\times 100$	B_{16}	B_{26}	Min.(D_{ij})	$\frac{B_{16}}{\min(D_{ij})}\times 10^4$	$\frac{B_{26}}{\min(D_{ij})}\times 10^4$
1	0	0	0.5	2.57	2.54	1.2	3.51	1.27	7588	4.6	1.6
			0.75	3.33	3.30	0.9	126	25.9	49379	25	52.4
			1.0	5.48	5.41	1.2	80.1	80.1	62882	12.7	12.7
			1.5	13.14	12.91	1.8	94.0	65.6	66256	14.2	9.8
			2.0	21.95	21.6	1.6	80.0	80.0	62881	12.7	12.7
1	1	0	0.5	1.97	1.95	1.0	129.9	33.9	28783	45.0	11.75
			0.75	2.15	2.11	1.8	126.4	25.9	49379	25.0	52.0
			1.0	2.74	2.70	1.4	80.0	80.0	62882	12.7	12.7
			1.5	4.52	4.46	1.3	5.9	143.8	36716	1.6	38.0
			2.0	7.9	7.8	1.2	34.0	129.0	25468	13.3	50.6
0	0	1	0.5	8.93	8.34	2.27	137.0	7.6	41817	32.7	1.8
			0.75	9.82	9.57	2.54	114.1	42.4	56951	20.0	7.4
			1.0	12.37	12.11	2.1	80.1	80.1	62882	12.7	12.7
			1.5	20.82	20.68	0.5	27.1	125.6	46879	5.7	26.0
			2.0	34.14	33.8	1.0	6.65	138.3	41423	1.5	33.0
1	0	1	0.5	2.41	2.38	1.25	3.51	1.27	7588	4.6	1.6
			0.75	3.05	2.99	1.96	126.4	25.9	49379	25.3	5.0
			1.0	4.72	4.58	2.96	38.0	132.0	25482	15.9	51.8
			1.5	10.1	9.92	1.7	25.4	141.3	31219	8.1	45.0
			2.0	16.51	16.25	1.57	91.0	64.3	65139	13.9	9.8

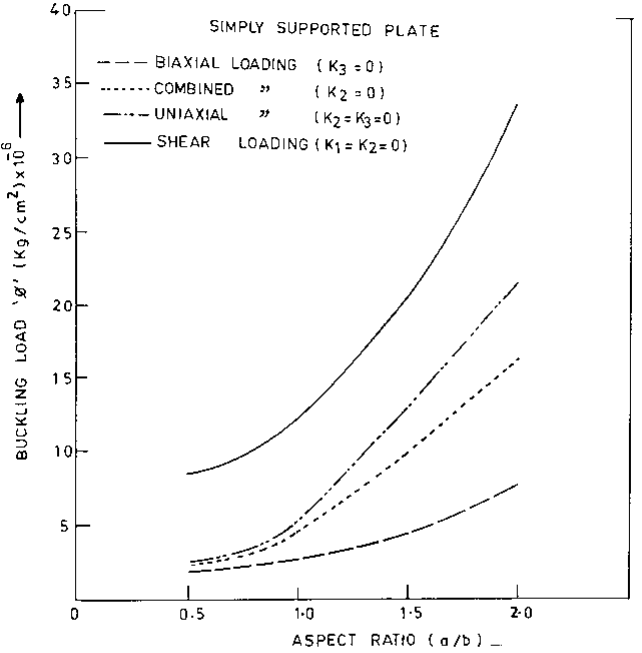


FIG. 3.1 VARIATION OF BUCKLING LOAD WITH ASPECT RATIO.

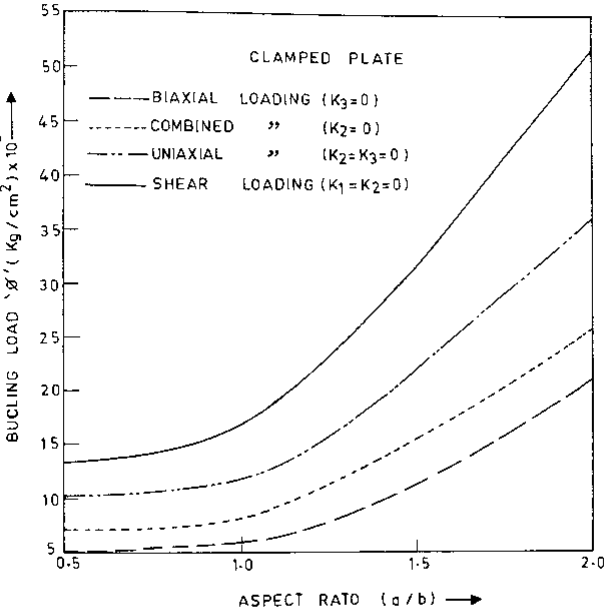


FIG.3.2 VARIATION OF BUCKLING LOAD WITH ASPECT RATIO .

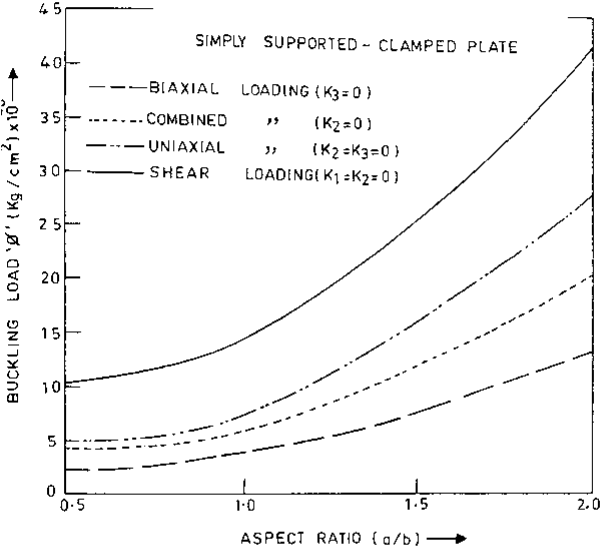


FIG.3.3 VARIATION OF BUCKLING LOAD WITH ASPECT RATIO.

CHAPTER IV

OPTIMIZATION OF HYBRID LAMINATED PLATES - II

Hybrids laminates are constructed by stacking, layers of different material properties. Various combinations of the constituents can be chosen to obtain desired properties of the laminate, as per requirements. Hybrid laminates can possess intermediate properties/cost levels. It is possible for example, to enhance the modulus, strength and fatigue performance of glass fibre reinforced, plastics by inclusion of C-fibres, or the impact energy of C-fibre reinforced plastics by inclusion of glass or aramid fibres.

In this chapter, carbon-glass-epoxy composite hybrid laminates have been considered. Each type of fibres, have their own advantages and disadvantages, regarding their cost, weight and strength. Elastic modulus of C-fibres is about five times than modulus of glass-fibres, while cost is about ten times. Density of glass-fibres is about 1.25 times of C-fibres. Thus to carry a specified buckling load by a hybrid laminate, one needs an optimum ratio of the constituents. An attempt is made to show that considerable cost can be saved by using carbon-glass hybrid laminate to carry loads higher than that are possible from glass hybrid laminates. It is also studied that how the percentage contributions of carbon and glass fibres vary with increase in buckling load.

4.1 OPTIMIZATION:

The objective function of the present study, is to minimize overall cost of the hybrid laminate for a specified buckling load. Total cost is the sum of the cost of glass and carbon fibres and matrix material (epoxy). Reducing overall cost is equivalent to using minimum carbon fibres, as the carbon is costliest component. Design variables are fibre volume fractions and fibre orientations of the layers of a laminate. Because of the conclusions of Chapter III, here also fibres in all the layers are assumed to be oriented in the same directions.

4.2 PROBLEM FORMULATION:

Let C be the total cost of hybrid laminated plate. Then objective function is,

Minimize $C(\theta_i, V_{fi})$

subject to following constraints:

1. Constraints on the fibre orientation

$$0^\circ \leq |\theta^\circ| \leq 90^\circ$$

2. Constraints on minimum individual percentage thickness of a layer,

$$t_i \geq t_{min}$$

t_{min} is taken as 6.25 percent from the practical considerations of the thickness of the prepex available.

3. Constraints on volume fractions,

$$V_{fmin} \leq V_{fi} \leq V_{fmax}$$

From practical considerations

$$V_{fmin} = 10 \text{ percent}$$
$$V_{fmax} = 70 \text{ percent}$$

4. Constraints for decoupling

$$\mathcal{C} > 0$$
$$R_1 > 0$$
$$R_2 > 0$$

5. Constraints on buckling load to be carried,

$$\text{Load} = (\text{Load})_{\text{specified}}$$

4.3 NUMERICAL COMPUTATIONS:

Numerical computations were carried out on carbon-epoxy composite system. The numerical values used in computation [14] are given below:

Material	Property			
	Elastic Modulus (E) (kg/cm^2)	Density (ρ) gm/cm^3	Non-dimensional Cost = $\frac{\text{Cost}}{\text{Cost of Epoxy}}$	Poisson's Ratio (ν)
Glass Fibre	7.32×10^5	2.54	2.3	0.25
Carbon Fibre	4.0×10^6	2.0	23	0.21
Epoxy	3.3×10^4	1.7	1	0.3

Firstly computations were carried out to maximize the buckling load ϕ defined as,

$$\phi = \frac{a^2 \lambda_{cr}}{t^3}$$

for glass-hybrid, six layered antisymmetric angle ply laminate, with total fibre content equivalent to 65 percent volume fraction of homogeneous laminate. This gives the maximum load carrying capacity of glass hybrid laminates as due to practical difficulties, volume fraction hardly exceeds 65 percent. Let this load be called $(Load)_G$. Now if carbon fibres are also used with glass to make a hybrid laminate buckling strength can be increased. Computations were carried out on six layered, carbon/carbon/glass (C/C/G) hybrid laminates and carbon hybrid-laminates. The objective function was the minimization of total cost for a specified load. Computations were made for various specified loads, starting from $(load)_G$ to two times, three times of $(Load)_G$ and so on.

Values of cost were non-dimensioned, by dividing them with cost of epoxy.

Let C be the total non-dimensional cost of hybrid laminate. C_i and ρ_i be the non-dimensional cost and density of the fibres, in the i-th layer. Let C_m and ρ_m be the cost and density of epoxy.

$$\text{Cost of the fibres} = 2 \left[\sum_{i=1}^{3} \rho_i (V_{fi}\, t_i)\, C_i \right] \qquad (4.1a)$$

$$\text{Cost of the matrix} = 2 \left[\left(\frac{1}{2} - \sum_{i=1}^{3} V_{fi}\, t_i \right) \rho_m\, C_m \right] \qquad (4.1b)$$

Total cost is now given by,

$$C = 2\left[\sum_{i=1}^{3} \rho_i V_{fi} t_i C_i + \left(\frac{1}{2} - \sum_{i=1}^{3} V_{fi} t_i\right) \rho_m C_m\right] \quad (4.1c)$$

Contribution of carbon and glass contents for (C/C/G) laminate can be given by,

$$\text{C-by volume} = \frac{V_{f1} t_1 + V_{f2} t_2}{\sum_{i=1}^{3} t_i} \quad (4.2)$$

$$\text{glass by volume} = \frac{V_{f3} t_3}{\sum_{i=1}^{3} t_i} \quad (4.3)$$

Weight of the fibres is given by,

$$w_f = 2\left[\sum_{i=1}^{3} \rho_i (V_{fi} t_i)\right] \quad (4.4a)$$

Weight of the matrix,

$$w_m = 2\left[\left(\frac{1}{2} - \sum_{i=1}^{3} (V_{fi} t_i)\right) \rho_m\right] \quad (4.4b)$$

Total weight of the laminate is:

$$w = 2\left[\sum_{i=1}^{3} \rho_i (V_{fi} t_i) + \left(\frac{1}{2} - \sum_{i=1}^{3} V_{fi} t_i\right) \rho_m\right] \quad (4.4c)$$

$$\text{Carbon by weight} = \frac{\rho_1 V_{f1} t_1 + \rho_2 V_{f2} t_2}{w} \quad (4.5)$$

$$\text{Glass by weight} = \frac{\rho_3 V_{f3} t_3}{3\ w} \quad (4.6)$$

4.4 RESULTS AND DISCUSSION:

The objective of the present study is, to compare the overall cost between carbon-glass hybrid and carbon hybrid laminates, for same specified load. Buckling load was maximized for glass-hybrid laminate with total fibre content, equivalent to 65 percent volume fraction homogeneous glass laminate. Calling this load as $(\text{Load})_G$, this load is taken as reference load, λ_{cr}. Loads above this value are possible only either by carbon hybrid or by carbon-glass hybrid. For such intermediate loads, optimization study is made to minimize cost at a specified load, for both carbon hybrid and carbon-glass hybrid laminates.

For the six-layered antisymmetric angle ply laminate considered, possible combinations of carbon and glass fibre layers are:

(a) Carbon/Carbon/Glass (C/C/G), (b) Carbon/Glass/Glass (C/G/G), (c) Glass/Carbon/Glass (G/C/G), (d) Glass/Glass/Carbon (G/G/C) and (e) Glass/Carbon/Carbon (G/C/C).

Out of these combinations, optimization results for the present analysis, could be obtained only for (C/C/G) and G/G/C. In other cases, it was found that decoupling is not possible. For complete decoupling of a laminate $\hat{\gamma}$, R_2 and R_1 must be positive, eqn. (2.25). These terms depend on material properties and fibre orientations of the laminae. Various volume fractions and fibre orientations combinations were tried, and it was found that in all the cases, either $\hat{\gamma}$ or thickness ratio R_1

comes out to be negative. Expression for R_1 eqn. (2.22) is,

$$R_1 = -\sqrt{1+\gamma} + \sqrt{1+\gamma(1-\alpha_{16})} - r_{16}$$

For the cases under discussion, it was found that,

$$-\sqrt{1+\gamma} > \sqrt{1+\gamma(1-\alpha_{16})} - r_{16}$$

making R_1 negative, and hence decoupling could not be obtained. In case of G/G/C, though decoupling is possible, it gives very narrow range of intermediate loads, and hence does not make much sense. In this chapter, study is made on C/C/G hybrid laminate.

Optimization results for (C/C/G) antisymmetric hybrid laminate are presented in Table 4.1. Overall cost is minimized at specified load. Computation were made for various specified loads, starting from $(Load)_G$ and the increasing it tillg possible for C/C/G hybrid laminate. Combined shear and axial loading and aspect ratio 1.5 has been considered for simply-supported and clamped boundary conditions.

It is observed from Table 4.1 that as specified load is increased, optimum fibre volume fractions also increase gradually. However, in all the cases, the volume fraction in middle carbon layer (V_{f2}) is always greater than the volume fraction of outermost layer (V_{f2}). When load is increased to four times, V_{f2} reaches the upper limit on fibre volume fractions (i.e. 70 percent). Since volume fraction in this layer now can not be exceeded, the buckling strength of C/C/G hybrid laminate can ~~be~~ not be increased further.

From the study of Table 4.1, it is observed that, with increase in buckling load, thicknesses of the layers containing carbon (t_1 and t_2) increase while that of containing glass (t_3), decrease. However, t_2 is always greater than t_1 and t_2.

Cost of C/C/G hybrid for the same specified buckling strength, are compared in Table 4.1. At load equal to $(Load)_C$, 23 percent cost can be saved for simply supported plate by using C/C/G hybrid. However, as the load is increased percentage difference in cost reduces and at load, four times that of $(Load)_C$, no considerable saving (only 3 percent) can be obtained. Hence C/C/G hybrid is verym much for lower range of intermediate loads. It is also seen that almost same results have been obtained in the clamped boundary conditions. However, weight of C/C/G hybrid is more and percentage difference in weight also reduced gradually with increase in load, from 12 percent to 5 percent in case of simply supported and from 11.6 percent to 4.1 percent in case of clamped conditions.

Fig. 4.1 and Fig. 4.2 show the comparison between cost and also weight for C/C/G and carbon hybrid laminates. It is observed that cost vary almost linearly with load. At low loads difference in cost in the two types of laminates is considerable, but as load is increased, cost of C/C/G approaches the cost of carbon hybrid.

However, weights do not vary linearly and curves for C/C/G and carbon hybrids are almost parallel. Comparing Fig. 4.1 and Fig. 4.2, it is observed that trends for the weight and cost, are more or less similar for both the boundary conditions i.e., simply supported and clamped.

Table 4.2 shows, the variation of glass and carbon contents with load. Fig. 4.3 and Fig. 4.4 are the plots between load and percentage fibre content. It is observed that glass content, both volume wise and weight wise, decreases very slowly (in curve it is almost a horizontal straight line). However, carbon content increases very fast and at load four times that of $(Load)_G$ its contribution is about 50 percent. Trends for the variation of each fibre contents by volume and by weight is same. Here also results are almost same for both the boundary conditions.

4.5 CONCLUSIONS:

Following conclusions have been obtained from the study made in this chapter:

1. Wide range of intermediate buckling loads upto four times that of glass hybrid, can be obtained, using carbon/carbon/glass hybrid laminates.
2. Optimum volume fraction and percentage thickness are always higher in middle layer, than that of other layers.
3. Decoupling is not possible for C/G/G, G/C/G and G/C/C combinations.
4. For lower range of buckling loads, as much as 23-25 percent, cost can be saved using C/C/G hybrid as compared to carbon hybrid laminates. However, at very high values of load, cost of C/C/G laminates approaches that of carbon hybrid laminates.

5. Cost of C/C/G hybrid laminates varies almost linearly with load.

6. Percentage carbon content in C/C/G laminate increases linearly with increase in load upto the load, three times that of glass hybrid load. Percentage glass content decreases very slowly and percentage glass content vs. load is almost a horizontal line.

7. Optimization results are not sensitive to boundary conditions.

SCOPE FOR FURTHER EXTENSION:

This work can be extended for further research work as stated below:

- Free edge and other boundary conditions can be studied.
- In case of hybrid laminates made of two types of fibres, other objective functions, like minimization of weight for specified buckling load, specific load and specific weight can be considered.
- Detailed analysis for Glass/Carbon/Glass, Carbon/Glass/Glass etc. combinations can be done.
- The effect of transverse shear can be studied.

Table 4.1: Optimization Results for (Carbon/Carbon/Glass) Antisymmetric Laminate

Combined Shear and Axial Loading ($K_1 = K_3 = 1$, $K_2 = 0$)

$(LOAD)_G$ For Simply Supported = 2.34×10^6 kg/cm^2

$(LOAD)_G$ For Clamped = 36×10^5 kg/cm^2

AR = 1.5

LOAD / $(LOAD)_G$	C/C/G HYBRID				% Thickness					CARBON HYBRID		Percent Difference in	
	θ	V_{f1}	V_{f2}	V_{f3}	t_1	t_2	t_3	W^+ ①	C^+ ②	W^+ ③	C^+ ④	Wt. ①-③/③	Cost ④-②/④
						SIMPLY SUPPORTED							
1	24.6	0.34	0.51	0.24	6.5	6.25	37.25	7.2	2.8	6.42	3.64	12.0	23.0
2	46.6	0.42	0.52	0.26	7.4	8.3	34.3	7.88	5.5	7.3	6.8	7.94	19.1
3	36.0	0.64	0.66	0.35	10.7	15.5	23.7	8.17	8.7	7.6	9.1	6.9	4.3
4	46.1	0.68	0.7	0.62	13.5	24.0	12.5	8.7	12.6	8.5	13.0	5.0	3.0
						CLAMPED							
1	60.8	0.2	0.4	0.2	6.4	7.65	36.0	7.48	2.62	6.7	3.52	11.6	25.6
2	42.9	0.49	0.55	0.2	8.2	9.8	30.0	7.74	4.25	7.1	5.27	8.4	19.3
3	44.2	0.55	0.6	0.36	11.3	16.2	22.5	8.02	7.31	7.5	7.9	7.0	7.4
4	44.5	0.69	0.7	0.56	13.5	24.0	12.4	8.83	12.7	8.4	13.6	4.1	6.7

W^+ = Weight/Area (gm/cm^2) x 10

C^+ = Non Dimensional Cost

Table 4.2: Variation of Carbon/Glass Content with Load.

(Carbon/Carbon/Glass) Antisymmetric Laminate
Combined Shear and Axial Loading ($K_1 = K_3 = 1$, $K_2 = 0$)
$(LOAD)_G$ for simply supported = 2.34×10^6 kg/cm^2
$(LOAD)_G$ for clamped = 36×10^6 kg/cm^2
AR = 1.5

Boundary Condition	$\frac{Load}{(Load)_G}$	Percent Carbon by Volume	Percent Glass by Volume	Percent Carbon by Weight	Percent Glass by Weight
Simply Supported	1	11.0	17.9	14.1	29.1
	2	23.4	17.6	28.7	25.3
	3	34.4	16.4	40.0	24.2
	4	51.9	15.6	56.4	21.5
Clamped	1	8.3	16.6	11.2	24.1
	2	20.2	15.9	19.0	23.9
	3	31.5	14.5	37.1	22.6
	4	52.3	14.0	57.4	19.6

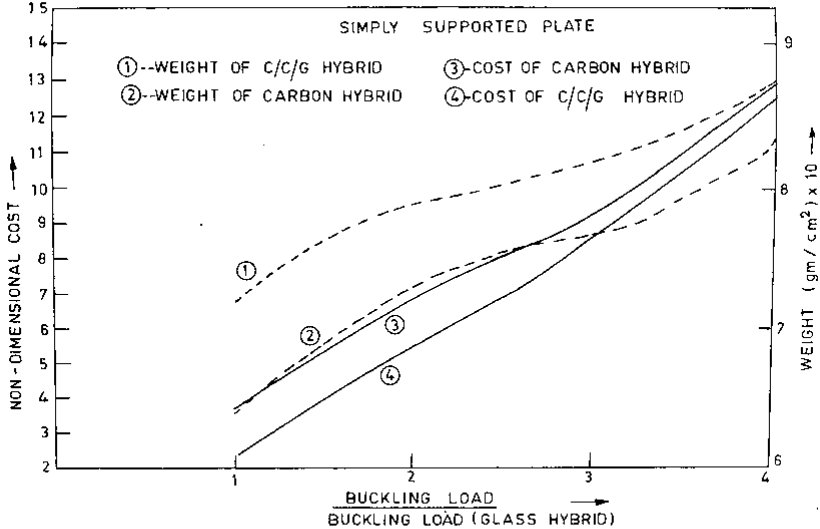


FIG. 4.1 COMPARISON OF COST AND WEIGHT BETWEEN C/C/G HYBRID AND CARBON HYBRID.

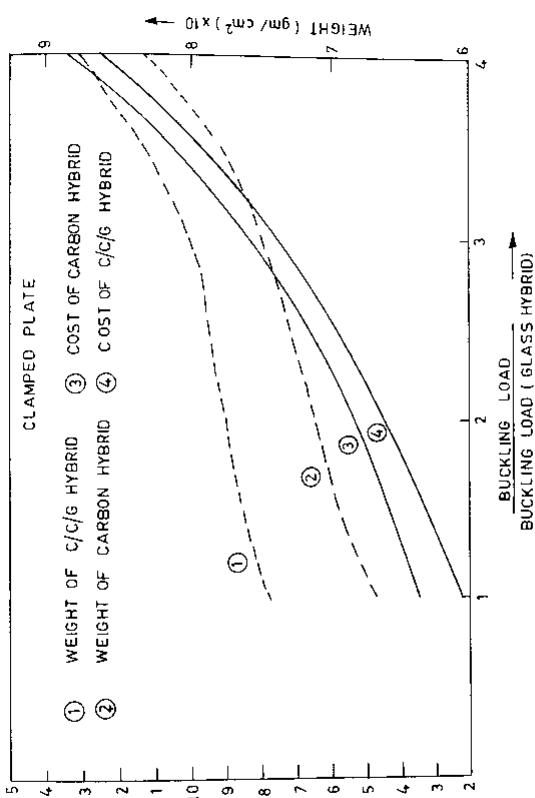

CLAMPED PLATE
① WEIGHT OF C/C/G HYBRID
② WEIGHT OF CARBON HYBRID
③ COST OF CARBON HYBRID
④ COST OF C/C/G HYBRID
NON-DIMENSIONAL COST
WEIGHT (gm/cm²) x10
BUCKLING LOAD / BUCKLING LOAD (GLASS HYBRID)

FIG.4.2 COMPARISON OF COST AND WEIGHT BETWEEN C/C/G HYBRID AND CARBON HYBRID.

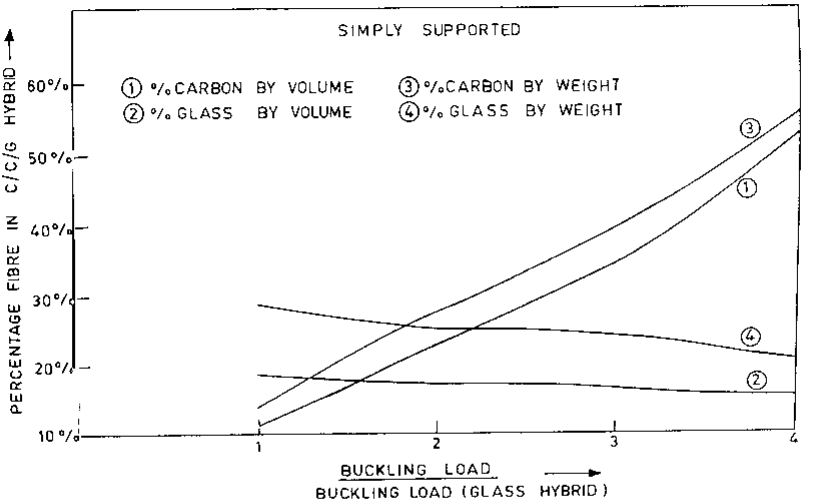


FIG. 4.3 VARIATION OF CARBON/GLASS CONTENT WITH LOAD IN C/C/G HYBRID.

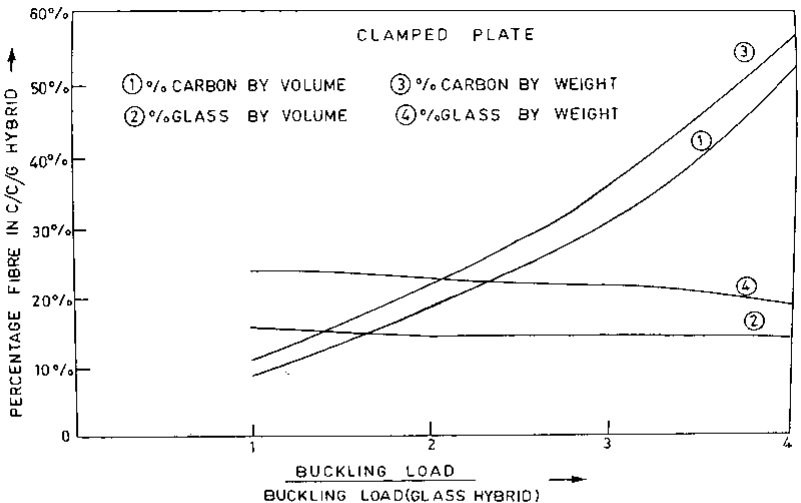


FIG.4.4 VARIATION OF CARBON /GLASS CONTENT WITH LOAD IN C/C/G HYBRID.

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